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Congruences involving Bernoulli and Euler numbers

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Abstract

Let $[x]$ be the integral part of x. Let $p>5$ be a prime. In the paper we mainly determine $\sum_{x=1}^{[p/4]}\frac{1}{x^k} \pmod{p^2}$, $\binom{p-1}{[p/4]} \pmod{p^3}$, $\sum_{k=1}^{p-1}\frac{2^k}{k} \pmod{p^3}$ and $\sum_{k=1}^{p-1}\frac{2^k}{k^2} \pmod{p^2}$ in terms of Euler and Bernoulli numbers. For example, we have

$$\sum_{x=1}^{[p/4]}\frac{1}{x^2}\equiv(-1)^{\frac{p-1}{2}}(8E_{p-3}-4E_{2p-4})+\frac{14}{3}pB_{p-3} \pmod{p^2},$$

where E_n is the nth Euler number and B_n is the nth Bernoulli number.

MSC: primary 11B68; secondary 11A07

Keywords: Congruence; Bernoulli number; Bernoulli polynomial; Euler number

1. Introduction

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0=1,\quad \sum_{k=0}^{n-1}\binom{n}{k}B_k=0\quad (n\geqslant 2)\quad\text{and}\quad B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}\quad (n\geqslant 0).$$

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The Euler numbers $\{E_n\}$ and Euler polynomials $\{E_n(x)\}$ are defined by

$$\frac{2e^t}{e^{2t}+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}\left(|t|<\frac{\pi}{2}\right)\quad\text{and}\quad\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad\big(|t|<\pi\big),$$

which are equivalent to (see [MOS])

$$E_0=1,\quad E_{2n-1}=0,\quad \sum_{r=0}^{n}\binom{2n}{2r}E_{2r}=0\quad(n\geqslant 1)$$

and

$$E_n(x)=\frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}(2x-1)^{n-r}E_r.$$

Let $[x]$ be the integral part of x. For a given prime p let $\mathbb{Z}_p$ denote the set of rational p-integers (those rational numbers whose denominator is not divisible by p). For $a\in\mathbb{Z}_p$ with $a\not\equiv 0 \pmod p$, as usual we define the Fermat quotient $q_p(a)=(a^{p-1}-1)/p$. In the paper we establish some congruences involving Bernoulli and Euler numbers. In particular, in $\mathbb{Z}_p$ we have

$$\sum_{\frac{p}{4}<k<\frac{p}{2}}\frac{1}{k}\equiv q_p(2)-p\left(\frac{1}{2}q_p(2)^2+(-1)^{\frac{p-1}{2}}(E_{2p-4}-2E_{p-3})\right)+\frac{1}{3}p^2q_p(2)^3\pmod{p^3},$$

$$(-1)^{[\frac{p}{4}]}\binom{p-1}{[\frac{p}{4}]}\equiv 1+3pq_p(2)+p^2\big(3q_p(2)^2-(-1)^{\frac{p-1}{2}}E_{p-3}\big)\pmod{p^3},$$

$$\sum_{\substack{1\leqslant k<p\\4\mid k+p}}\frac{1}{k}\equiv\frac{1}{4}q_p(2)-\frac{1}{8}pq_p(2)^2+\frac{1}{12}p^2q_p(2)^3-\frac{7}{192}p^2B_{p-3}\pmod{p^3},$$

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\equiv -2q_p(2)-\frac{7}{12}p^2B_{p-3}\pmod{p^3},$$

$$\sum_{k=1}^{p-1}\frac{2^k}{k^2}\equiv -q_p(2)^2+p\left(\frac{2}{3}q_p(2)^3+\frac{7}{6}B_{p-3}\right)\pmod{p^2},$$

where p is a prime greater than 5.

In addition to the above notation, we also use throughout this paper the following notation: $\mathbb{Z}$—the set of integers, $\mathbb{N}$—the set of positive integers, $\{x\}$—the fractional part of x, $\varphi(n)$—Euler's totient function.

2. Basic lemmas

We begin with a useful identity involving Bernoulli polynomials.

Lemma 2.1. *Let $p, m \in \mathbb{N}$ and $k, r \in \mathbb{Z}$ with $k \geqslant 0$. Then*

$$\sum_{\substack{x=0\\x\equiv r \pmod m}}^{p-1} x^k = \frac{m^k}{k+1}\left(B_{k+1}\left(\frac{p}{m}+\left\{\frac{r-p}{m}\right\}\right)-B_{k+1}\left(\left\{\frac{r}{m}\right\}\right)\right).$$

In the case $m=1$ Lemma 2.1 is well known. See [MOS,IR]. Lemma 2.1 was established by the author in 1991. A proof is given in [S4], and a generalization was published by author's brother Z.W. Sun [Su]. From [S2, Lemma 2.3] and [IR, Proposition 15.2.4, p. 238] we have

Lemma 2.2. *Suppose that $k, p \in \mathbb{N}$ with $p > 1$. If $x, y \in \mathbb{Z}_p$, then $pB_k(x) \in \mathbb{Z}_p$ and $(B_k(x) - B_k(y))/k \in \mathbb{Z}_p$. If p is an odd prime such that $p-1 \nmid k$, then $B_k(x)/k \in \mathbb{Z}_p$.*

Lemma 2.3. *(See [MOS].) Let x and y be variables and $n \in \mathbb{N}$. Then*

(i) $B_{2n+1} = 0$.
(ii) $B_n(1-x) = (-1)^n B_n(x)$.
(iii) $B_n(x+y) = \sum_{r=0}^{n}\binom{n}{r}B_{n-r}(y)x^r$.
(iv) $E_{n-1}(x) = \frac{2^n}{n}(B_n(\frac{x+1}{2}) - B_n(\frac{x}{2}))$.

Lemma 2.4. *(See [MOS,GS].) Let $n \in \mathbb{N}$. Then*

$$B_{2n}\left(\frac{1}{4}\right) = B_{2n}\left(\frac{3}{4}\right) = \frac{2-2^{2n}}{4^{2n}}B_{2n}, \quad B_{2n}\left(\frac{1}{3}\right) = B_{2n}\left(\frac{2}{3}\right) = \frac{3-3^{2n}}{2\cdot 3^{2n}}B_{2n}$$

and

$$B_{2n}\left(\frac{1}{6}\right) = B_{2n}\left(\frac{5}{6}\right) = \frac{(2-2^{2n})(3-3^{2n})}{2\cdot 6^{2n}}B_{2n}.$$

Lemma 2.5. *For any nonnegative integer n we have*

$$E_{2n} = -4^{2n+1}\frac{B_{2n+1}(\frac{1}{4})}{2n+1}.$$

Proof. It is well known that $E_{2n} = 2^{2n}E_{2n}(\frac{1}{2})$. Thus applying Lemma 2.3 we see that

$$\begin{aligned}E_{2n} &= 2^{2n}E_{2n+1-1}\left(\frac{1}{2}\right) = 2^{2n}\cdot\frac{2^{2n+1}}{2n+1}\left(B_{2n+1}\left(\frac{3}{4}\right)-B_{2n+1}\left(\frac{1}{4}\right)\right)\\&= \frac{2^{4n+1}}{2n+1}\left(-B_{2n+1}\left(\frac{1}{4}\right)-B_{2n+1}\left(\frac{1}{4}\right)\right) = -\frac{2^{4n+2}}{2n+1}B_{2n+1}\left(\frac{1}{4}\right).\end{aligned}$$

This proves the lemma. $\square$

From [S3, Corollary 3.1 and Theorem 4.2] we have:

Lemma 2.6. *Let p be an odd prime, $x \in \mathbb{Z}_p$ and $k, b \in \mathbb{N}$ with $p-1 \nmid b$. Then*

$$\frac{B_{k(p-1)+b}(x)}{k(p-1)+b} \equiv \frac{B_b(x)}{b} \pmod p \quad \text{for } b \geqslant 2$$

and

$$\frac{B_{k(p-1)+b}(x)}{k(p-1)+b} \equiv k\frac{B_{p-1+b}(x)}{p-1+b} - (k-1)\frac{B_b(x)}{b} \pmod{p^2} \quad \text{for } b > 2.$$

Lemma 2.7. *Let $p > 3$ be a prime, $r \in \mathbb{Z}$ and $k, m \in \mathbb{N}$ with $k < p-3$ and $p \nmid m$. Then*

$$\begin{aligned}\sum_{\substack{x=1\\ x\equiv r \pmod m}}^{p-1} \frac{1}{x^k} &\equiv \frac{B_{\varphi(p^3)-k+1}(\{\frac{r-p}{m}\}) - B_{\varphi(p^3)-k+1}(\{\frac{r}{m}\})}{m^k(\varphi(p^3)-k+1)} \\ &\quad + \frac{kp}{m^{k+1}}\left(\frac{B_{2p-2-k}(\{\frac{r-p}{m}\})}{2p-2-k} - 2\frac{B_{p-1-k}(\{\frac{r-p}{m}\})}{p-1-k}\right) \\ &\quad - \frac{k(k+1)p^2}{2(k+2)m^{k+2}} B_{p-2-k}\left(\left\{\frac{r-p}{m}\right\}\right) \pmod{p^3}.\end{aligned}$$

Proof. From Lemmas 2.1, 2.3(iii) and Euler's theorem we see that

$$\begin{aligned}\sum_{\substack{x=1\\ x\equiv r \pmod m}}^{p-1} \frac{1}{x^k} &\equiv \sum_{\substack{x=0\\ x\equiv r \pmod m}}^{p-1} x^{\varphi(p^3)-k} \\ &= \frac{m^{\varphi(p^3)-k}}{\varphi(p^3)-k+1}\left(B_{\varphi(p^3)-k+1}\left(\frac{p}{m} + \left\{\frac{r-p}{m}\right\}\right) - B_{\varphi(p^3)-k+1}\left(\left\{\frac{r}{m}\right\}\right)\right) \\ &= \frac{m^{\varphi(p^3)-k}}{\varphi(p^3)-k+1}\left\{\sum_{j=0}^{\varphi(p^3)-k+1}\binom{\varphi(p^3)-k+1}{j}\frac{p^j}{m^j}B_{\varphi(p^3)-k+1-j}\left(\left\{\frac{r-p}{m}\right\}\right)\right. \\ &\quad \left. - B_{\varphi(p^3)-k+1}\left(\left\{\frac{r}{m}\right\}\right)\right\} \\ &= m^{\varphi(p^3)-k}\left\{\frac{B_{\varphi(p^3)-k+1}(\{\frac{r-p}{m}\}) - B_{\varphi(p^3)-k+1}(\{\frac{r}{m}\})}{\varphi(p^3)-k+1} + \frac{p}{m}B_{\varphi(p^3)-k}\left(\left\{\frac{r-p}{m}\right\}\right)\right. \\ &\quad + \frac{\varphi(p^3)-k}{2}\cdot\frac{p^2}{m^2}B_{\varphi(p^3)-k-1}\left(\left\{\frac{r-p}{m}\right\}\right) \\ &\quad \left. + \sum_{j=3}^{\varphi(p^3)-k+1}\frac{p^{j-3}}{j}\binom{\varphi(p^3)-k}{j-1}\frac{p^3}{m^j}B_{\varphi(p^3)-k+1-j}\left(\left\{\frac{r-p}{m}\right\}\right)\right\} \pmod{p^3}.\end{aligned}$$

As $k < p-3$ we have $p-1 \nmid \varphi(p^3)-k-2$ and so $B_{\varphi(p^3)-k-2}(\{\frac{r-p}{m}\}) \in \mathbb{Z}_p$ by Lemma 2.2. For $j \geqslant 4$ we have $p^{j-3}/j \equiv 0 \pmod p$. Thus $\frac{p^{j-3}}{j} B_{\varphi(p^3)-k+1-j}(\{\frac{r-p}{m}\}) \in \mathbb{Z}_p$ for $j \geqslant 3$. Hence, by the above we obtain

$$m^k \sum_{\substack{x=1\\ x\equiv r \pmod m}}^{p-1} \frac{1}{x^k} \equiv \frac{B_{\varphi(p^3)-k+1}(\{\frac{r-p}{m}\}) - B_{\varphi(p^3)-k+1}(\{\frac{r}{m}\})}{\varphi(p^3)-k+1} + \frac{p}{m} B_{\varphi(p^3)-k}\left(\left\{\frac{r-p}{m}\right\}\right)$$
$$+ \frac{\varphi(p^3)-k}{2} \cdot \frac{p^2}{m^2} B_{\varphi(p^3)-k-1}\left(\left\{\frac{r-p}{m}\right\}\right) \pmod{p^3}.$$

From Lemma 2.6 we see that

$$\frac{B_{\varphi(p^3)-k-1}(\{\frac{r-p}{m}\})}{\varphi(p^3)-k-1} = \frac{B_{(p^2-1)(p-1)+p-2-k}(\{\frac{r-p}{m}\})}{(p^2-1)(p-1)+p-2-k} \equiv \frac{B_{p-2-k}(\{\frac{r-p}{m}\})}{p-2-k} \pmod p$$

and

$$\frac{B_{\varphi(p^3)-k}(\{\frac{r-p}{m}\})}{\varphi(p^3)-k} = \frac{B_{(p^2-1)(p-1)+p-1-k}(\{\frac{r-p}{m}\})}{(p^2-1)(p-1)+p-1-k}$$
$$\equiv (p^2-1)\frac{B_{2p-2-k}(\{\frac{r-p}{m}\})}{2p-2-k} - (p^2-2)\frac{B_{p-1-k}(\{\frac{r-p}{m}\})}{p-1-k} \pmod{p^2}.$$

Thus

$$B_{\varphi(p^3)-k-1}\left(\left\{\frac{r-p}{m}\right\}\right) \equiv \frac{k+1}{k+2} B_{p-2-k}\left(\left\{\frac{r-p}{m}\right\}\right) \pmod p$$

and

$$B_{\varphi(p^3)-k}\left(\left\{\frac{r-p}{m}\right\}\right) \equiv k\frac{B_{2p-2-k}(\{\frac{r-p}{m}\})}{2p-2-k} - 2k\frac{B_{p-1-k}(\{\frac{r-p}{m}\})}{p-1-k} \pmod{p^2}.$$

Now putting all the above together we obtain the result. □

Lemma 2.8. *Let p be an odd prime, $a \in \mathbb{Z}_p$, $a \not\equiv 0 \pmod p$, $n \in \mathbb{N}$ and $p > n+1$. Then*

$$\frac{a^{\varphi(p^n)}-1}{p^n} \equiv \sum_{s=1}^{n} \frac{(-1)^{s-1}}{s} p^{s-1} q_p(a)^s \pmod{p^n}.$$

Proof. As $p > n+1 \geqslant 2$ we see that $p^{s-1}/s! \in \mathbb{Z}_p$ and $p^{s-1}/s! \equiv 0 \pmod p$ for $s \geqslant 2$. Thus,

$$\frac{a^{\varphi(p^n)}-1}{p^n} = \frac{(1+pq_p(a))^{p^{n-1}}-1}{p^n} = \frac{1}{p^n}\sum_{s=1}^{p^{n-1}} \binom{p^{n-1}}{s} p^s q_p(a)^s$$

$$= q_p(a) + \sum_{s=2}^{p^{n-1}} \left(p^{n-1}-1\right)\cdots\left(p^{n-1}-s+1\right)\cdot \frac{p^{s-1}}{s!} q_p(a)^s$$

$$\equiv q_p(a) + \sum_{s=2}^{p^{n-1}} (-1)(-2)\cdots(-s+1)\cdot \frac{p^{s-1}}{s!} q_p(a)^s$$

$$= \sum_{s=1}^{p^{n-1}} \frac{(-1)^{s-1}}{s} p^{s-1} q_p(a)^s \pmod{p^n}.$$

As $p > n+1$ we see that $p^{s-n-1}/s \in \mathbb{Z}_p$ for $s \geqslant n+1$. Thus for $s \geqslant n+1$ we have

$$\frac{(-1)^{s-1}}{s} p^{s-1} q_p(a)^s = (-1)^{s-1}\cdot \frac{p^{s-n-1}}{s}\cdot q_p(a)^s \cdot p^n \equiv 0 \pmod{p^n}.$$

Now putting the above together we obtain the result. □

Lemma 2.9. *Let p be an odd prime and $k \in \{0, 1, \dots, p-1\}$. Then*

$$(-1)^k \binom{p-1}{k} \equiv 1 - p\sum_{i=1}^k \frac{1}{i} + \frac{p^2}{2}\left\{\left(\sum_{i=1}^k \frac{1}{i}\right)^2 - \sum_{i=1}^k \frac{1}{i^2}\right\}$$
$$- \frac{p^3}{6}\left\{\left(\sum_{i=1}^k \frac{1}{i}\right)^3 - 3\left(\sum_{i=1}^k \frac{1}{i}\right)\left(\sum_{i=1}^k \frac{1}{i^2}\right) + 2\sum_{i=1}^k \frac{1}{i^3}\right\} \pmod{p^4}.$$

Proof. For $k = 0, 1, 2$ it is easy to verify the result. Now assume $k \geqslant 3$. Clearly

$$\binom{p-1}{k} = \frac{(p-1)(p-2)\cdots(p-k)}{k!}$$
$$= \frac{1}{k!}\left\{p^k + \cdots + p^3 \sum_{1\leqslant i<j<l\leqslant k} \frac{(-1)(-2)\cdots(-k)}{(-i)(-j)(-l)}\right.$$
$$\left. + p^2 \sum_{1\leqslant i<j\leqslant k} \frac{(-1)(-2)\cdots(-k)}{(-i)(-j)} + p\sum_{i=1}^k \frac{(-1)(-2)\cdots(-k)}{-i} + (-1)^k k!\right\}$$
$$\equiv (-1)^k \left(-p^3 \sum_{1\leqslant i<j<l\leqslant k} \frac{1}{ijl} + p^2 \sum_{1\leqslant i<j\leqslant k} \frac{1}{ij} - p\sum_{i=1}^k \frac{1}{i} + 1\right) \pmod{p^4}.$$

Observe that

$$\left(\sum_{i=1}^k \frac{1}{i}\right)^2 = \sum_{i=1}^k \frac{1}{i^2} + 2\sum_{1\leqslant i<j\leqslant k} \frac{1}{ij}$$

and

$$\left(\sum_{i=1}^{k}\frac{1}{i}\right)^3=\sum_{1\leqslant i,j,l\leqslant k}\frac{1}{ijl}=6\sum_{1\leqslant i<j<l\leqslant k}\frac{1}{ijl}+3\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{1}{i^2j}+\sum_{i=1}^{k}\frac{1}{i^3}$$
$$=6\sum_{1\leqslant i<j<l\leqslant k}\frac{1}{ijl}+3\left(\sum_{j=1}^{k}\frac{1}{j}\right)\left(\sum_{i=1}^{k}\frac{1}{i^2}\right)-3\sum_{i=1}^{k}\frac{1}{i^3}+\sum_{i=1}^{k}\frac{1}{i^3}.$$

We then have

$$\sum_{1\leqslant i<j\leqslant k}\frac{1}{ij}=\frac{1}{2}\left\{\left(\sum_{i=1}^{k}\frac{1}{i}\right)^2-\sum_{i=1}^{k}\frac{1}{i^2}\right\}$$

and

$$\sum_{1\leqslant i<j<l\leqslant k}\frac{1}{ijl}=\frac{1}{6}\left\{\left(\sum_{i=1}^{k}\frac{1}{i}\right)^3-3\left(\sum_{i=1}^{k}\frac{1}{i}\right)\left(\sum_{i=1}^{k}\frac{1}{i^2}\right)+2\sum_{i=1}^{k}\frac{1}{i^3}\right\}.$$

Now putting all the above together we obtain the result. $\square$

We remark that the congruence for $\binom{p-1}{k}\pmod{p^3}$ was given by Lehmer in [L, p. 360].

3. Congruences for $\sum_{1\leqslant x<p,\, m\mid x-p}\frac{1}{x^k}$ $(m=3,4,6)$ and $\sum_{x=1}^{[p/4]}\frac{1}{x^k}$

Theorem 3.1. *Let $p>3$ be a prime. Then*

(i) $$\sum_{\substack{k=1\\ k\equiv p \pmod 3}}^{p-1}\frac{1}{k}\equiv\frac{1}{2}q_p(3)-\frac{1}{4}pq_p(3)^2+\frac{1}{6}p^2q_p(3)^3-\frac{p^2}{81}B_{p-3}\pmod{p^3}.$$

(ii) $$\sum_{\substack{k=1\\ k\equiv p \pmod 4}}^{p-1}\frac{1}{k}\equiv\frac{3}{4}q_p(2)-\frac{3}{8}pq_p(2)^2+\frac{1}{4}p^2q_p(2)^3-\frac{p^2}{192}B_{p-3}\pmod{p^3}.$$

(iii) $$\sum_{\substack{k=1\\ k\equiv p \pmod 6}}^{p-1}\frac{1}{k}\equiv\frac{1}{3}q_p(2)+\frac{1}{4}q_p(3)-p\left(\frac{1}{6}q_p(2)^2+\frac{1}{8}q_p(3)^2\right)$$
$$+p^2\left(\frac{1}{9}q_p(2)^3+\frac{1}{12}q_p(3)^3-\frac{1}{648}B_{p-3}\right)\pmod{p^3}.$$

Proof. Note that $B_n(0)=B_n$ and $B_{2n+1}=0$ for $n\in\mathbb{N}$. Taking $k=1$ and $r=p$ in Lemma 2.7 we see that if $m\in\mathbb{N}$ and $p\nmid m$, then

$$\sum_{\substack{k=1\\ k\equiv p \pmod m}}^{p-1}\frac{1}{k}\equiv\frac{B_{p^2(p-1)}-B_{p^2(p-1)}(\{\frac{p}{m}\})}{mp^2(p-1)}-\frac{p^2}{3m^3}B_{p-3}\pmod{p^3}.$$

As $B_{2n}(x) = B_{2n}(1-x)$, for $m = 3, 4, 6$ we have $B_{2n}(\{\frac{p}{m}\}) = B_{2n}(\frac{1}{m})$. Since $pB_{p^2(p-1)} \equiv p-1 \pmod{p^3}$ by [S2, Corollary 4.1], using Lemmas 2.4, 2.8 and Euler's theorem we see that

$$\begin{aligned}\frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{3})}{p^2(p-1)} &= \left(1 - \frac{3 - 3^{p^2(p-1)}}{2 \cdot 3^{p^2(p-1)}}\right)\frac{B_{p^2(p-1)}}{p^2(p-1)} \\ &= \frac{3}{2 \cdot 3^{p^2(p-1)}} \cdot \frac{3^{p^2(p-1)} - 1}{p^3} \cdot \frac{pB_{p^2(p-1)}}{p-1} \equiv \frac{3}{2} \cdot \frac{3^{p^2(p-1)} - 1}{p^3} \\ &\equiv \frac{3}{2}\left(q_p(3) - \frac{1}{2}pq_p(3)^2 + \frac{1}{3}p^2q_p(3)^3\right) \pmod{p^3}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{4})}{p^2(p-1)} &= \left(1 - \frac{2 - 2^{p^2(p-1)}}{4^{p^2(p-1)}}\right)\frac{B_{p^2(p-1)}}{p^2(p-1)} \\ &= \frac{2^{p^2(p-1)} + 2}{4^{p^2(p-1)}} \cdot \frac{2^{p^2(p-1)} - 1}{p^3} \cdot \frac{pB_{p^2(p-1)}}{p-1} \equiv 3 \cdot \frac{2^{p^2(p-1)} - 1}{p^3} \\ &\equiv 3\left(q_p(2) - \frac{1}{2}pq_p(2)^2 + \frac{1}{3}p^2q_p(2)^3\right) \pmod{p^3}\end{aligned}$$

and

$$\begin{aligned}&\frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{6})}{p^2(p-1)} \\ &= \left(1 - \frac{(2 - 2^{p^2(p-1)})(3 - 3^{p^2(p-1)})}{2 \cdot 6^{p^2(p-1)}}\right)\frac{B_{p^2(p-1)}}{p^2(p-1)} \\ &= \frac{(2^{p^2(p-1)} - 1)(3^{p^2(p-1)} - 1) + 4(2^{p^2(p-1)} - 1) + 3(3^{p^2(p-1)} - 1)}{2 \cdot 6^{p^2(p-1)} \cdot p^3} \cdot \frac{pB_{p^2(p-1)}}{p-1} \\ &\equiv 2 \cdot \frac{2^{p^2(p-1)} - 1}{p^3} + \frac{3}{2} \cdot \frac{3^{p^2(p-1)} - 1}{p^3} \\ &\equiv 2\left(q_p(2) - \frac{1}{2}pq_p(2)^2 + \frac{1}{3}p^2q_p(2)^3\right) + \frac{3}{2}\left(q_p(3) - \frac{1}{2}pq_p(3)^2 + \frac{1}{3}p^2q_p(3)^3\right) \pmod{p^3}.\end{aligned}$$

Now combining all the above we obtain the result. □

Corollary 3.1. *Let $p > 3$ be a prime. Then*

$$\sum_{\substack{k=1 \\ k \equiv -p \pmod 4}}^{p-1} \frac{1}{k} \equiv \frac{1}{4}q_p(2) - \frac{1}{8}pq_p(2)^2 + \frac{1}{12}p^2q_p(2)^3 - \frac{7}{192}p^2B_{p-3} \pmod{p^3}.$$

Proof. Using [S3, Remark 5.3] we know that

$$\sum_{\substack{k=1\\k\equiv -p \pmod 4}}^{p-1}\frac{1}{k}+\sum_{\substack{k=1\\k\equiv p \pmod 4}}^{p-1}\frac{1}{k}$$
$$=\sum_{\substack{k=1\\2\nmid k}}^{p-1}\frac{1}{k}\equiv q_p(2)-\frac{1}{2}pq_p(2)^2+\frac{1}{3}p^2q_p(2)^3-\frac{1}{24}p^2B_{p-3}\pmod{p^3}.$$

Thus applying Theorem 3.1(ii) we deduce the result. □

Remark 3.1. Let $m\in\{3,4,6\}$. In 1938 E. Lehmer [L] obtained the congruences for

$$\sum_{\substack{k=1\\k\equiv p \pmod m}}^{p-1}\frac{1}{k}\pmod{p^2}.$$

Using the formulas for $\sum_{k\equiv r \pmod m}\binom{p}{k}$, in [S1] the author gave congruences for

$$\sum_{\substack{k=1\\k\equiv r \pmod m}}^{p-1}\frac{1}{k}\pmod p.$$

Corollary 3.2. *Let $p>3$ be a prime. Then*

$$\sum_{\substack{k=1\\k\equiv -p \pmod 3}}^{\frac{p-1}{2}}\frac{1}{k}\equiv -\frac{2}{3}q_p(2)+\frac{1}{2}q_p(3)+p\left(\frac{1}{3}q_p(2)^2-\frac{1}{4}q_p(3)^2\right)$$
$$+p^2\left(-\frac{2}{9}q_p(2)^3+\frac{1}{6}q_p(3)^3-\frac{7}{324}B_{p-3}\right)\pmod{p^3}.$$

Proof. Clearly

$$\sum_{\substack{k=1\\k\equiv -p \pmod 3}}^{\frac{p-1}{2}}\frac{1}{k}=2\sum_{\substack{k=1\\2k\equiv p+3 \pmod 6}}^{\frac{p-1}{2}}\frac{1}{2k}=2\sum_{\substack{k=1\\k\equiv p+3 \pmod 6}}^{p-1}\frac{1}{k}$$
$$=2\left(\sum_{\substack{k=1\\k\equiv p \pmod 3}}^{p-1}\frac{1}{k}-\sum_{\substack{k=1\\k\equiv p \pmod 6}}^{p-1}\frac{1}{k}\right).$$

Thus appealing to Theorem 3.1 we obtain the result. □

Theorem 3.2. *Let $p > 3$ be a prime. Then*

$$\sum_{1\leqslant k<\frac{p}{4}} \frac{1}{k} \equiv -3q_p(2) + p\left(\frac{3}{2}q_p(2)^2 + (-1)^{\frac{p-1}{2}}(E_{2p-4} - 2E_{p-3})\right) - p^2\left(q_p(2)^3 + \frac{7}{12}B_{p-3}\right) \pmod{p^3}$$

and

$$\sum_{\frac{p}{4}<k<\frac{p}{2}} \frac{1}{k} \equiv q_p(2) - p\left(\frac{1}{2}q_p(2)^2 + (-1)^{\frac{p-1}{2}}(E_{2p-4} - 2E_{p-3})\right) + \frac{1}{3}p^2 q_p(2)^3 \pmod{p^3}.$$

Proof. Taking $k = 1$, $r = 0$ and $m = 4$ in Lemma 2.7 we find

$$\sum_{\substack{x=1\\4\mid x}}^{p-1} \frac{1}{x} \equiv \frac{B_{p^2(p-1)}(\{\frac{-p}{4}\}) - B_{p^2(p-1)}}{4p^2(p-1)} + \frac{p}{16}\left(\frac{B_{2p-3}(\{\frac{-p}{4}\})}{2p-3} - 2\frac{B_{p-2}(\{\frac{-p}{4}\})}{p-2}\right) - \frac{p^2}{192}B_{p-3}\left(\left\{\frac{-p}{4}\right\}\right) \pmod{p^3}.$$

As $B_n(\frac{3}{4}) = (-1)^n B_n(\frac{1}{4})$ we then have

$$\sum_{1\leqslant k<\frac{p}{4}} \frac{1}{k} = 4\sum_{\substack{x=1\\4\mid x}}^{p-1} \frac{1}{x} \equiv \frac{B_{p^2(p-1)}(\frac{1}{4}) - B_{p^2(p-1)}}{p^2(p-1)} + (-1)^{\frac{p+1}{2}}\frac{p}{4}\left(\frac{B_{2p-3}(\frac{1}{4})}{2p-3} - 2\frac{B_{p-2}(\frac{1}{4})}{p-2}\right) - \frac{p^2}{48}B_{p-3}\left(\frac{1}{4}\right) \pmod{p^3}.$$

From the proof of Theorem 3.1 we know that

$$\frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{4})}{p^2(p-1)} \equiv 3q_p(2) - \frac{3}{2}pq_p(2)^2 + p^2 q_p(2)^3 \pmod{p^3}.$$

By Lemmas 2.5 and 2.6 we have

$$E_{2p-4} = -4^{2p-3}\frac{B_{2p-3}(\frac{1}{4})}{2p-3} \equiv -4^{p-2}\frac{B_{p-2}(\frac{1}{4})}{p-2} = E_{p-3} \pmod{p}. \tag{3.1}$$

Observe that $a^{s(p-1)} = (1 + pq_p(a))^s \equiv 1 + spq_p(a) \pmod{p^2}$ for $a, s \in \mathbb{Z}$ with $p \nmid a$. We then have

$$\frac{1}{4^{2p-2}} \equiv 1 - 2pq_p(4) \pmod{p^2} \quad\text{and}\quad \frac{1}{4^{p-1}} \equiv 1 - pq_p(4) \pmod{p^2}.$$

Hence

$$\begin{aligned}
&-\frac{1}{4}\left(\frac{B_{2p-3}(\frac{1}{4})}{2p-3}-2\frac{B_{p-2}(\frac{1}{4})}{p-2}\right)\\
&=\frac{E_{2p-4}}{4^{2p-2}}-2\frac{E_{p-3}}{4^{p-1}}\equiv\big(1-2pq_p(4)\big)E_{2p-4}-2\big(1-pq_p(4)\big)E_{p-3}\\
&=E_{2p-4}-2E_{p-3}-2pq_p(4)(E_{2p-4}-E_{p-3})\\
&\equiv E_{2p-4}-2E_{p-3}\pmod{p^2}.
\end{aligned}$$

On the other hand, by Lemma 2.4 we have

$$B_{p-3}\left(\frac{1}{4}\right)=\frac{2-2^{p-3}}{4^{p-3}}B_{p-3}=\frac{32-4\cdot 2^{p-1}}{4^{p-1}}B_{p-3}\equiv 28B_{p-3}\pmod{p}.$$

Thus combining the above we obtain

$$\begin{aligned}
\sum_{1\leqslant k<\frac{p}{4}}\frac{1}{k}\equiv{}&-3q_p(2)+\frac{3}{2}pq_p(2)^2-p^2q_p(2)^3+(-1)^{\frac{p-1}{2}}p(E_{2p-4}-2E_{p-3})\\
&-\frac{p^2}{48}\cdot 28B_{p-3}\pmod{p^3}.
\end{aligned}$$

From [S3, Theorem 5.2] we know that

$$\sum_{1\leqslant k<\frac{p}{2}}\frac{1}{k}\equiv -2q_p(2)+pq_p(2)^2-\frac{2}{3}p^2q_p(2)^3-\frac{7}{12}p^2B_{p-3}\pmod{p^3}.$$

Observe that

$$\sum_{\frac{p}{4}<k<\frac{p}{2}}\frac{1}{k}=\sum_{1\leqslant k<\frac{p}{2}}\frac{1}{k}-\sum_{1\leqslant k<\frac{p}{4}}\frac{1}{k}.$$

We then obtain the remaining result. □

Remark 3.2. For any prime $p>3$, the congruence $\sum_{k=1}^{[\frac{p}{4}]}\frac{1}{k}\equiv -3q_p(2)\pmod{p}$ was first established by Lerch (see [D]), and a simple proof concerning the formula for $\sum_{4\mid k}\binom{p}{k}$ was given by the author in [S1].

Corollary 3.3. *Let $p>3$ be a prime. Then*

$$\sum_{1\leqslant k<\frac{p}{4}}\frac{1}{k}\equiv -3q_p(2)+\frac{3}{2}pq_p(2)^2-(-1)^{\frac{p-1}{2}}pE_{p-3}\pmod{p^2}$$

and

$$\sum_{\frac{p}{4}<k<\frac{p}{2}} \frac{1}{k} \equiv q_p(2) - \frac{1}{2} p q_p(2)^2 + (-1)^{\frac{p-1}{2}} p E_{p-3} \pmod{p^2}.$$

Lemma 3.1. *Let $p > 5$ be a prime, $r \in \mathbb{Z}$, $k, m \in \mathbb{N}$, $p \nmid m$ and $1 < k < p-3$. Then*

$$\begin{aligned} m^k & \sum_{\substack{x=1 \\ x \equiv r \pmod m}}^{p-1} \frac{1}{x^k} \\ & \equiv \frac{B_{2p-1-k}(\{\frac{r}{m}\}) - B_{2p-1-k}(\{\frac{r-p}{m}\})}{2p-1-k} - 2\frac{B_{p-k}(\{\frac{r}{m}\}) - B_{p-k}(\{\frac{r-p}{m}\})}{p-k} \\ & \quad + \frac{kp}{m(k+1)} B_{p-1-k}\left(\left\{\frac{r-p}{m}\right\}\right) \pmod{p^2}. \end{aligned}$$

Proof. From Lemma 2.6 we see that if $x \in \mathbb{Z}_p$, then

$$\frac{B_{2p-2-k}(x)}{2p-2-k} \equiv \frac{B_{p-1-k}(x)}{p-1-k} \pmod p$$

and

$$\begin{aligned} \frac{B_{\varphi(p^3)-k+1}(x)}{\varphi(p^3)-k+1} &= \frac{B_{(p^2-1)(p-1)+p-k}(x)}{(p^2-1)(p-1)+p-k} \\ &\equiv (p^2-1)\frac{B_{2p-1-k}(x)}{2p-1-k} - (p^2-2)\frac{B_{p-k}(x)}{p-k} \\ &\equiv -\frac{B_{2p-1-k}(x)}{2p-1-k} + 2\frac{B_{p-k}(x)}{p-k} \pmod{p^2}. \end{aligned}$$

This together with Lemma 2.7 gives the result. □

Putting $r = 0, p$ in Lemma 3.1 and noting that $B_{2n+1} = 0$ $(n \geqslant 1)$ we deduce the following result.

Theorem 3.3. *Let $p > 5$ be a prime, $k, m \in \mathbb{N}$, $p \nmid m$ and $1 < k < p-3$. Then*

$$\begin{aligned} m^k & \sum_{\substack{x=1 \\ x \equiv p \pmod m}}^{p-1} \frac{1}{x^k} \\ & \equiv \begin{cases} \frac{B_{2p-1-k}(\{\frac{p}{m}\}) - B_{2p-1-k}}{2p-1-k} - 2\frac{B_{p-k}(\{\frac{p}{m}\}) - B_{p-k}}{p-k} \pmod{p^2} & \text{if } 2 \nmid k, \\ \frac{B_{2p-1-k}(\{\frac{p}{m}\})}{2p-1-k} - 2\frac{B_{p-k}(\{\frac{p}{m}\})}{p-k} + \frac{kp}{m(k+1)} B_{p-1-k} \pmod{p^2} & \text{if } 2 \mid k. \end{cases} \end{aligned}$$

and

$$\sum_{x=1}^{[p/m]}\frac{1}{x^k}\equiv\begin{cases}\frac{B_{2p-1-k}-B_{2p-1-k}(\{\frac{-p}{m}\})}{2p-1-k}-2\frac{B_{p-k}-B_{p-k}(\{\frac{-p}{m}\})}{p-k}\\ \quad+\frac{kp}{m(k+1)}B_{p-1-k}(\{\frac{-p}{m}\}) \pmod{p^2} & \text{if } 2\nmid k,\\ -\frac{B_{2p-1-k}(\{\frac{-p}{m}\})}{2p-1-k}+2\frac{B_{p-k}(\{\frac{-p}{m}\})}{p-k}\\ \quad+\frac{kp}{m(k+1)}B_{p-1-k}(\{\frac{-p}{m}\}) \pmod{p^2} & \text{if } 2\mid k.\end{cases}$$

Corollary 3.4. *Let $p>5$ be a prime and $k\in\{3,5,\dots,p-4\}$. Then*

(i) $$\sum_{x=1}^{[p/3]}\frac{1}{x^k}\equiv-\frac{3^k-3}{2k}B_{p-k}\pmod p,$$

(ii) $$\sum_{x=1}^{[p/4]}\frac{1}{x^k}\equiv-\frac{2^{2k-1}-2^{k-1}-1}{k}B_{p-k}\pmod p,$$

(iii) $$\sum_{x=1}^{[p/6]}\frac{1}{x^k}\equiv-\frac{(2^k-1)(3^k-1)-2}{2k}B_{p-k}\pmod p,$$

(iv) $$\sum_{\frac{p}{6}<x<\frac{p}{4}}\frac{1}{x^k}\equiv\frac{(2^k-1)(3^k-2^k-1)}{2k}B_{p-k}\pmod p,$$

(v) $$\sum_{\frac{p}{4}<x<\frac{p}{3}}\frac{1}{x^k}\equiv\frac{2^{2k}-2^k-3^k+1}{2k}B_{p-k}\pmod p.$$

Proof. From Lemma 2.6 and Theorem 3.3 we see that for $m=3,4,6$,

$$\sum_{x=1}^{[p/m]}\frac{1}{x^k}\equiv-\frac{B_{p-k}-B_{p-k}(\{\frac{-p}{m}\})}{p-k}=\frac{B_{p-k}(\frac{1}{m})-B_{p-k}}{p-k}\pmod p.$$

Now applying Lemma 2.4 we deduce (i)–(iii). (iv) follows from (ii) and (iii), and (v) follows from (i) and (ii). □

Theorem 3.4. *Let $p>5$ be a prime and $k\in\{3,5,\dots,p-4\}$. Then*

(i) $$\sum_{\substack{x=1\\ x\equiv p\ (\mathrm{mod}\ 3)}}^{p-1}\frac{1}{x^k}\equiv\frac{3^{k-1}-1}{2\cdot 3^{k-1}}\left(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\right)\pmod{p^2},$$

(ii) $$\sum_{\substack{x=1\\ x\equiv p\ (\mathrm{mod}\ 4)}}^{p-1}\frac{1}{x^k}\equiv\frac{2^{2k-1}-2^{k-1}-1}{2^{2k}}\left(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\right)\pmod{p^2},$$

(iii) $$\sum_{\substack{x=1\\ x\equiv p\ (\mathrm{mod}\ 6)}}^{p-1}\frac{1}{x^k}\equiv\frac{(2^k-1)(3^k-1)-2}{2\cdot 6^k}\left(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\right)\pmod{p^2}.$$

Proof. Let $m \in \{3,4,6\}$. As $B_{2n}(1-x) = B_{2n}(x)$, we see that $B_{2n}(\{\frac{p}{m}\}) = B_{2n}(\frac{1}{m})$. Hence, applying Theorem 3.3 we have

$$m^k \sum_{\substack{x=1\\ x\equiv p \pmod m}}^{p-1} \frac{1}{x^k} \equiv \frac{B_{2p-1-k}(\frac{1}{m}) - B_{2p-1-k}}{2p-1-k} - 2\frac{B_{p-k}(\frac{1}{m}) - B_{p-k}}{p-k} \pmod{p^2}. \quad (3.2)$$

By Lemma 2.4 we have $B_{2n}(\frac{1}{3}) - B_{2n} = \frac{1}{2}(3^{1-2n}-3)B_{2n}$. Note that $3^{2-2p} \equiv 2\cdot 3^{1-p} - 1 \pmod{p^2}$. By the above we obtain

$$\begin{aligned}
3^k \sum_{\substack{x=1\\ x\equiv p \pmod 3}}^{p-1} \frac{1}{x^k} &\equiv \frac{3^{1-(2p-1-k)}-3}{2}\cdot\frac{B_{2p-1-k}}{2p-1-k} - 2\cdot\frac{3^{1-(p-k)}-3}{2}\cdot\frac{B_{p-k}}{p-k}\\
&\equiv \frac{3^k(2\cdot 3^{1-p}-1)-3}{2}\cdot\frac{B_{2p-1-k}}{2p-1-k} - \left(3^{k+1-p}-3\right)\frac{B_{p-k}}{p-k}\\
&= 3^{k+1-p}\left(\frac{B_{2p-1-k}}{2p-1-k} - \frac{B_{p-k}}{p-k}\right) - \frac{3^k+3}{2}\cdot\frac{B_{2p-1-k}}{2p-1-k} + 3\frac{B_{p-k}}{p-k}\\
&\equiv 3^k\left(\frac{B_{2p-1-k}}{2p-1-k} - \frac{B_{p-k}}{p-k}\right) - \frac{3^k+3}{2}\cdot\frac{B_{2p-1-k}}{2p-1-k} + 3\frac{B_{p-k}}{p-k}\\
&= \frac{3^k-3}{2}\cdot\frac{B_{2p-1-k}}{2p-1-k} - \left(3^k-3\right)\frac{B_{p-k}}{p-k} \pmod{p^2}.
\end{aligned}$$

This proves (i). Now we consider (ii). From Lemma 2.4 we know that $B_{2n}(\frac{1}{4}) - B_{2n} = (2^{1-4n} - 2^{-2n} - 1)B_{2n}$. Observe that $2^{s(p-1)} = (1+pq_p(2))^s \equiv 1 + spq_p(2) \pmod{p^2}$. Using (3.2) we see that

$$\begin{aligned}
4^k \sum_{\substack{x=1\\ x\equiv p \pmod 4}}^{p-1} \frac{1}{x^k} &\equiv \left(2^{2k-1-4(p-1)} - 2^{k-1-2(p-1)} - 1\right)\frac{B_{2p-1-k}}{2p-1-k}\\
&\quad - 2\left(2^{2k-1-2(p-1)} - 2^{k-1-(p-1)} - 1\right)\frac{B_{p-k}}{p-k}\\
&\equiv \left(2^{2k-1}\left(1-4pq_p(2)\right) - 2^{k-1}\left(1-2pq_p(2)\right) - 1\right)\frac{B_{2p-1-k}}{2p-1-k}\\
&\quad - 2\left(2^{2k-1}\left(1-2pq_p(2)\right) - 2^{k-1}\left(1-pq_p(2)\right) - 1\right)\frac{B_{p-k}}{p-k}\\
&= \left(2^k - 2^{2k+1}\right)q_p(2)p\left(\frac{B_{2p-1-k}}{2p-1-k} - \frac{B_{p-k}}{p-k}\right)\\
&\quad + \left(2^{2k-1} - 2^{k-1} - 1\right)\frac{B_{2p-1-k}}{2p-1-k} - 2\left(2^{2k-1} - 2^{k-1} - 1\right)\frac{B_{p-k}}{p-k}\\
&\equiv \left(2^{2k-1} - 2^{k-1} - 1\right)\left(\frac{B_{2p-1-k}}{2p-1-k} - 2\frac{B_{p-k}}{p-k}\right) \pmod{p^2}.
\end{aligned}$$

This proves (ii). Finally we consider (iii). As $B_{2n}(\frac{1}{6}) = \frac{1}{2}(2^{1-2n}-1)(3^{1-2n}-1)B_{2n}$, by (3.2) we have

$$\begin{aligned}
&6^k \sum_{\substack{x=1\\ x\equiv p \pmod 6}}^{p-1} \frac{1}{x^k}\\
&\equiv \frac{1}{2}\big(\big(2^{k-2(p-1)}-1\big)\big(3^{k-2(p-1)}-1\big)-2\big)\frac{B_{2p-1-k}}{2p-1-k}\\
&\quad -\big(\big(2^{k-(p-1)}-1\big)\big(3^{k-(p-1)}-1\big)-2\big)\frac{B_{p-k}}{p-k}\\
&\equiv \frac{1}{2}\big\{\big(2^k\big(1-2pq_p(2)\big)-1\big)\big(3^k\big(1-2pq_p(3)\big)-1\big)-2\big\}\frac{B_{2p-1-k}}{2p-1-k}\\
&\quad -\big\{\big(2^k\big(1-pq_p(2)\big)-1\big)\big(3^k\big(1-pq_p(3)\big)-1\big)-2\big\}\frac{B_{p-k}}{p-k}\\
&\equiv \left(\frac{(2^k-1)(3^k-1)-2}{2}-2^k\big(3^k-1\big)pq_p(2)-3^k\big(2^k-1\big)pq_p(3)\right)\frac{B_{2p-1-k}}{2p-1-k}\\
&\quad -\big(\big(2^k-1\big)\big(3^k-1\big)-2-2^k\big(3^k-1\big)pq_p(2)-3^k\big(2^k-1\big)pq_p(3)\big)\frac{B_{p-k}}{p-k}\\
&= \frac{(2^k-1)(3^k-1)-2}{2}\left(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\right)\\
&\quad -\big(2^k\big(3^k-1\big)q_p(2)+3^k\big(2^k-1\big)q_p(3)\big)p\left(\frac{B_{2p-1-k}}{2p-1-k}-\frac{B_{p-k}}{p-k}\right)\\
&\equiv \frac{(2^k-1)(3^k-1)-2}{2}\left(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\right) \pmod{p^2}.
\end{aligned}$$

This proves (iii) and hence the theorem is proved. □

Corollary 3.5. *Let $p > 5$ be a prime and $k \in \{3, 5, \dots, p-4\}$. Then*

$$\sum_{\substack{x=1\\ x\equiv p \pmod 3}}^{p-1} \frac{1}{x^k} \equiv \frac{3^{k-1}-1}{2k\cdot 3^{k-1}} B_{p-k} \pmod p,$$

$$\sum_{\substack{x=1\\ x\equiv p \pmod 4}}^{p-1} \frac{1}{x^k} \equiv \frac{2^{2k-1}-2^{k-1}-1}{k\cdot 2^{2k}} B_{p-k} \pmod p,$$

$$\sum_{\substack{x=1\\ x\equiv p \pmod 6}}^{p-1} \frac{1}{x^k} \equiv \frac{(2^k-1)(3^k-1)-2}{2k\cdot 6^k} B_{p-k} \pmod p.$$

Corollary 3.6. *Let $p > 5$ be a prime and $k \in \{3, 5, \ldots, p-4\}$. Then*

$$\sum_{\substack{x=1\\x\equiv -p \pmod 4}}^{p-1} \frac{1}{x^k} \equiv \frac{2^{2k-1} - 3\cdot 2^{k-1} + 1}{2^{2k}} \left(\frac{B_{2p-1-k}}{2p-1-k} - 2\frac{B_{p-k}}{p-k} \right) \pmod{p^2}.$$

Proof. From [S3, Theorems 5.1 and 5.2] we see that

$$\sum_{\substack{x=1\\2\nmid x}}^{p-1} \frac{1}{x^k} = \sum_{x=1}^{p-1} \frac{1}{x^k} - \frac{1}{2^k} \sum_{x=1}^{(p-1)/2} \frac{1}{x^k} \equiv -\frac{2^k-2}{2^k} \left(2\frac{B_{p-k}}{p-k} - \frac{B_{2p-1-k}}{2p-1-k} \right) \pmod{p^2}.$$

Thus

$$\sum_{\substack{x=1\\x\equiv p \pmod 4}}^{p-1} \frac{1}{x^k} + \sum_{\substack{x=1\\x\equiv -p \pmod 4}}^{p-1} \frac{1}{x^k} \equiv \frac{2^k-2}{2^k} \left(\frac{B_{2p-1-k}}{2p-1-k} - 2\frac{B_{p-k}}{p-k} \right) \pmod{p^2}.$$

Now applying Theorem 3.4 we deduce the result. □

Theorem 3.5. *Let $p > 5$ be a prime and $k \in \{2, 4, \ldots, p-5\}$. Then*

$$\sum_{1\leqslant x<\frac{p}{4}} \frac{1}{x^k} \equiv (-1)^{\frac{p-1}{2}} 4^{k-1} (2E_{p-1-k} - E_{2p-2-k}) + \frac{2^{k-2}(2^{k+1}-1)k}{k+1} pB_{p-1-k} \pmod{p^2}$$

and

$$\sum_{\frac{p}{4}<x<\frac{p}{2}} \frac{1}{x^k} \equiv (-1)^{\frac{p+1}{2}} 4^{k-1} (2E_{p-1-k} - E_{2p-2-k}) - \frac{k(2^{k-1}-1)(2^{k+1}-1)}{2(k+1)} pB_{p-1-k} \pmod{p^2}.$$

Proof. As $B_n(\frac{3}{4}) = (-1)^n B_n(\frac{1}{4})$, putting $m = 4$ in Theorem 3.3 we see that

$$\sum_{x=1}^{[p/4]} \frac{1}{x^k} \equiv (-1)^{\frac{p-1}{2}} \left(\frac{B_{2p-1-k}(\frac{1}{4})}{2p-1-k} - 2\frac{B_{p-k}(\frac{1}{4})}{p-k} \right) + \frac{kp}{4(k+1)} B_{p-1-k}\left(\frac{1}{4}\right) \pmod{p^2}.$$

From Lemmas 2.4 and 2.5 we have

$$\frac{B_{2p-1-k}(\frac{1}{4})}{2p-1-k} = -\frac{4^{k-1}E_{2p-2-k}}{4^{2(p-1)}} \equiv -4^{k-1}\big(1-2pq_p(4)\big)E_{2p-2-k} \pmod{p^2},$$
$$\frac{B_{p-k}(\frac{1}{4})}{p-k} = -\frac{4^{k-1}E_{p-1-k}}{4^{p-1}} \equiv -4^{k-1}\big(1-pq_p(4)\big)E_{p-1-k} \pmod{p^2},$$
$$B_{p-1-k}\left(\frac{1}{4}\right) = \frac{2-2^{p-1-k}}{4^{p-1-k}}B_{p-1-k} \equiv 2^k\big(2^{k+1}-1\big)B_{p-1-k} \pmod{p}.$$

As $B_{2p-1-k}(\frac{1}{4})/(2p-1-k) \equiv B_{p-k}(\frac{1}{4})/(p-k) \pmod{p}$, we see that $E_{2p-2-k} \equiv E_{p-1-k} \pmod{p}$ and so

$$\begin{aligned}\sum_{x=1}^{[p/4]}\frac{1}{x^k} &\equiv (-1)^{\frac{p-1}{2}}\big(-4^{k-1}\big)\big\{\big(1-2pq_p(4)\big)E_{2p-2-k} - 2\big(1-pq_p(4)\big)E_{p-1-k}\big\} \\ &\quad + \frac{kp}{4(k+1)}\cdot 2^k\big(2^{k+1}-1\big)B_{p-1-k} \\ &\equiv (-1)^{\frac{p-1}{2}}4^{k-1}(2E_{p-1-k}-E_{2p-2-k}) + \frac{2^{k-2}(2^{k+1}-1)k}{k+1}pB_{p-1-k} \pmod{p^2}.\end{aligned}$$

By [S3, Corollary 5.2(a)], we have

$$\sum_{1\leqslant x<\frac{p}{2}}\frac{1}{x^k} \equiv \frac{k(2^{k+1}-1)}{2(k+1)}pB_{p-1-k} \pmod{p^2}.$$

Note that

$$\sum_{\frac{p}{4}<x<\frac{p}{2}}\frac{1}{x^k} = \sum_{1\leqslant x<\frac{p}{2}}\frac{1}{x^k} - \sum_{1\leqslant x<\frac{p}{4}}\frac{1}{x^k}.$$

By the above we obtain the remaining result. □

Corollary 3.7. *(Lehmer [L, (20)].) Let $p>5$ be a prime and $k\in\{2,4,\dots,p-5\}$. Then*

$$\sum_{x=1}^{[p/4]}\frac{1}{x^k} \equiv (-1)^{\frac{p-1}{2}}4^{k-1}E_{p-1-k} \pmod{p}.$$

Corollary 3.8. *Let $p>5$ be a prime. Then*

$$\sum_{x=1}^{[p/4]}\frac{1}{x^2} \equiv (-1)^{\frac{p-1}{2}}(8E_{p-3}-4E_{2p-4}) + \frac{14}{3}pB_{p-3} \pmod{p^2}.$$

Theorem 3.6. *Let $p > 5$ be a prime and $k \in \{3, 5, \ldots, p-4\}$. Then*

$$\sum_{1\leqslant x<\frac{p}{4}} \frac{1}{x^k} \equiv \left(2^{2k-1} - 2^{k-1} - 1\right)\left(2\frac{B_{p-k}}{p-k} - \frac{B_{2p-1-k}}{2p-1-k}\right) - (-1)^{\frac{p-1}{2}} 4^{k-1} kpE_{p-2-k} \pmod{p^2}$$

and

$$\sum_{\frac{p}{4}<x<\frac{p}{2}} \frac{1}{x^k} \equiv -\left(2^{2k-1} - 3\cdot 2^{k-1} + 1\right)\left(2\frac{B_{p-k}}{p-k} - \frac{B_{2p-1-k}}{2p-1-k}\right) + (-1)^{\frac{p-1}{2}} 4^{k-1} kpE_{p-2-k} \pmod{p^2}.$$

Proof. As $B_n(\frac{3}{4}) = (-1)^n B_n(\frac{1}{4})$, putting $m = 4$ in Theorem 3.3 we see that

$$\sum_{1\leqslant x<\frac{p}{4}} \frac{1}{x^k} \equiv \frac{B_{2p-1-k} - B_{2p-1-k}(\frac{1}{4})}{2p-1-k} - 2\frac{B_{p-k} - B_{p-k}(\frac{1}{4})}{p-k} + \frac{kp}{4(k+1)} \cdot (-1)^{\frac{p+1}{2}} B_{p-1-k}\left(\frac{1}{4}\right) \pmod{p^2}.$$

According to Lemmas 2.4 and 2.5 we have

$$B_{2n} - B_{2n}\left(\frac{1}{4}\right) = \left(1 + 2^{-2n} - 2^{1-4n}\right)B_{2n} \quad \text{and} \quad B_{2n+1}\left(\frac{1}{4}\right) = -\frac{2n+1}{4^{2n+1}}E_{2n}.$$

Thus,

$$\begin{aligned}
&\sum_{1\leqslant x<\frac{p}{4}} \frac{1}{x^k} \\
&\equiv \left(1 + 2^{k-1-2(p-1)} - 2^{2k-1-4(p-1)}\right)\frac{B_{2p-1-k}}{2p-1-k} \\
&\quad - 2\left(1 + 2^{k-1-(p-1)} - 2^{2k-1-2(p-1)}\right)\frac{B_{p-k}}{p-k} \\
&\quad + \frac{kp}{4(k+1)} \cdot (-1)^{\frac{p+1}{2}}\left(-4^{k-(p-1)}(p-1-k)\right)E_{p-2-k} \\
&\equiv \left(1 + 2^{k-1}\left(1 - 2pq_p(2)\right) - 2^{2k-1}\left(1 - 4pq_p(2)\right)\right)\frac{B_{2p-1-k}}{2p-1-k} \\
&\quad - 2\left(1 + 2^{k-1}\left(1 - pq_p(2)\right) - 2^{2k-1}\left(1 - 2pq_p(2)\right)\right)\frac{B_{p-k}}{p-k} + (-1)^{\frac{p+1}{2}} 4^{k-1} kpE_{p-2-k} \\
&= \left(1 + 2^{k-1} - 2^{2k-1}\right)\left(\frac{B_{2p-1-k}}{2p-1-k} - 2\frac{B_{p-k}}{p-k}\right)
\end{aligned}$$

$$+\big(2^{2k+1}-2^k\big)q_p(2)p\left(\frac{B_{2p-1-k}}{2p-1-k}-\frac{B_{p-k}}{p-k}\right)+(-1)^{\frac{p+1}{2}}4^{k-1}kpE_{p-2-k}$$
$$\equiv\big(1+2^{k-1}-2^{2k-1}\big)\left(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\right)+(-1)^{\frac{p+1}{2}}4^{k-1}kpE_{p-2-k}\pmod{p^2}.$$

From [S3, Theorem 5.2(b)] we have

$$\sum_{1\leqslant x<\frac{p}{2}}\frac{1}{x^k}\equiv\big(2^k-2\big)\left(2\frac{B_{p-k}}{p-k}-\frac{B_{2p-1-k}}{2p-1-k}\right)\pmod{p^2}.$$

Now combining the above we deduce the result. □

Theorem 3.7. *Let $p>5$ be a prime. Then*

$$\sum_{\frac{p}{2}<k<\frac{3p}{4}}\frac{1}{k}\equiv -q_p(2)+p\left(\frac{1}{2}q_p(2)^2+(-1)^{\frac{p-1}{2}}(6E_{p-3}-3E_{2p-4})\right)$$
$$-\frac{1}{3}p^2\big(q_p(2)^3+14B_{p-3}\big)\pmod{p^3}$$

and

$$\sum_{\frac{3p}{4}<k<p}\frac{1}{k}\equiv 3q_p(2)-p\left(\frac{3}{2}q_p(2)^2+(-1)^{\frac{p-1}{2}}(6E_{p-3}-3E_{2p-4})\right)$$
$$+p^2\left(q_p(2)^3+\frac{59}{12}B_{p-3}\right)\pmod{p^3}.$$

Proof. It is clear that

$$\sum_{1\leqslant k<\frac{p}{4}}\frac{1}{k}+\sum_{\frac{3p}{4}<k<p}\frac{1}{k}=\sum_{1\leqslant k<\frac{p}{4}}\left(\frac{1}{k}+\frac{1}{p-k}\right)=\sum_{1\leqslant k<\frac{p}{4}}\frac{p}{k(p-k)}$$
$$=p\sum_{1\leqslant k<\frac{p}{4}}\frac{p+k}{kp^2-k^3}\equiv p\sum_{1\leqslant k<\frac{p}{4}}\frac{p+k}{-k^3}$$
$$=-p^2\sum_{1\leqslant k<\frac{p}{4}}\frac{1}{k^3}-p\sum_{1\leqslant k<\frac{p}{4}}\frac{1}{k^2}\pmod{p^3}.$$

By Corollaries 3.4 and 3.8 we have

$$\sum_{1\leqslant k<\frac{p}{4}}\frac{1}{k^3}\equiv -9B_{p-3}\pmod{p}$$

and

$$\sum_{1\leqslant k<\frac{p}{4}}\frac{1}{k^2}\equiv 4(-1)^{\frac{p-1}{2}}(2E_{p-3}-E_{2p-4})+\frac{14}{3}pB_{p-3}\pmod{p^2}.$$

Thus

$$\begin{aligned}&\sum_{1\leqslant k<\frac{p}{4}}\frac{1}{k}+\sum_{\frac{3p}{4}<k<p}\frac{1}{k}\\&\quad\equiv -p^2(-9B_{p-3})-p\left(4(-1)^{\frac{p-1}{2}}(2E_{p-3}-E_{2p-4})+\frac{14}{3}pB_{p-3}\right)\\&\quad= -4(-1)^{\frac{p-1}{2}}(2E_{p-3}-E_{2p-4})p+\frac{13}{3}p^2B_{p-3}\pmod{p^3}.\end{aligned}$$

Hence applying Theorem 3.2 we obtain

$$\begin{aligned}\sum_{\frac{3p}{4}<k<p}\frac{1}{k}&\equiv -4(-1)^{\frac{p-1}{2}}(2E_{p-3}-E_{2p-4})p+\frac{13}{3}p^2B_{p-3}+3q_p(2)\\&\quad -p\left(\frac{3}{2}q_p(2)^2-(-1)^{\frac{p-1}{2}}(2E_{p-3}-E_{2p-4})\right)+p^2\left(q_p(2)^3+\frac{7}{12}B_{p-3}\right)\\&=3q_p(2)-p\left(\frac{3}{2}q_p(2)^2+(-1)^{\frac{p-1}{2}}(6E_{p-3}-3E_{2p-4})\right)\\&\quad +p^2\left(q_p(2)^3+\frac{59}{12}B_{p-3}\right)\pmod{p^3}.\end{aligned}$$

From [L, p. 353] or [S3, Theorem 5.1(a)] we have

$$\sum_{1\leqslant k<p}\frac{1}{k}\equiv -\frac{1}{3}p^2B_{p-3}\pmod{p^3}.$$

By [S3, Theorem 5.2(c)],

$$\sum_{1\leqslant k<\frac{p}{2}}\frac{1}{k}\equiv -2q_p(2)+pq_p(2)^2-\frac{2}{3}p^2q_p(2)^3-\frac{7}{12}p^2B_{p-3}\pmod{p^3}.$$

Thus

$$\begin{aligned}\sum_{\frac{p}{2}<k<p}\frac{1}{k}&=\sum_{1\leqslant k<p}\frac{1}{k}-\sum_{1\leqslant k<\frac{p}{2}}\frac{1}{k}\\&\equiv 2q_p(2)-pq_p(2)^2+\frac{2}{3}p^2q_p(2)^3+\frac{1}{4}p^2B_{p-3}\pmod{p^3}.\end{aligned}$$

Observing that

$$\sum_{\frac{p}{2}<k<\frac{3p}{4}} \frac{1}{k} = \sum_{\frac{p}{2}<k<p} \frac{1}{k} - \sum_{\frac{3p}{4}<k<p} \frac{1}{k}$$

and applying the above we obtain the remaining result. □

Corollary 3.9. *Let $p > 5$ be a prime. Then*

$$\sum_{\frac{p}{2}<k<\frac{3p}{4}} \frac{1}{k} \equiv -q_p(2) + \frac{1}{2}pq_p(2)^2 + 3(-1)^{\frac{p-1}{2}} pE_{p-3} \pmod{p^2}$$

and

$$\sum_{\frac{3p}{4}<k<p} \frac{1}{k} \equiv 3q_p(2) - \frac{3}{2}pq_p(2)^2 - 3(-1)^{\frac{p-1}{2}} pE_{p-3} \pmod{p^2}.$$

Theorem 3.8. *Let $p > 5$ be a prime. Then*

$$(-1)^{[\frac{p}{4}]}\binom{p-1}{[\frac{p}{4}]} \equiv 1 + 3pq_p(2) + p^2\big(3q_p(2)^2 - (-1)^{\frac{p-1}{2}} E_{p-3}\big) \pmod{p^3}$$

and

$$(-1)^{[\frac{p}{4}]}\binom{p-1}{[\frac{p}{4}]} \equiv 1 + 3pq_p(2) + p^2\big(3q_p(2)^2 - (-1)^{\frac{p-1}{2}}(2E_{p-3} - E_{2p-4})\big) + p^3\left(q_p(2)^3 - 3(-1)^{\frac{p-1}{2}} q_p(2)E_{p-3} + \frac{5}{4}B_{p-3}\right) \pmod{p^4}.$$

Proof. From Theorem 3.2 we have

$$\sum_{i=1}^{[p/4]} \frac{1}{i} \equiv -3q_p(2) + p\left(\frac{3}{2}q_p(2)^2 + (-1)^{\frac{p-1}{2}}(E_{2p-4} - 2E_{p-3})\right) - p^2\left(q_p(2)^3 + \frac{7}{12}B_{p-3}\right) \pmod{p^3}.$$

By Corollaries 3.3 and 3.8 we have

$$\sum_{i=1}^{[p/4]} \frac{1}{i} \equiv -3q_p(2) + \frac{3}{2}pq_p(2)^2 - (-1)^{\frac{p-1}{2}} pE_{p-3} \pmod{p^2}$$

and

$$\sum_{i=1}^{[p/4]} \frac{1}{i^2} \equiv (-1)^{\frac{p-1}{2}}(8E_{p-3} - 4E_{2p-4}) + \frac{14}{3}pB_{p-3} \pmod{p^2}.$$

From this we deduce

$$\left(\sum_{i=1}^{[p/4]}\frac{1}{i}\right)^2-\sum_{i=1}^{[p/4]}\frac{1}{i^2}\equiv 9q_p(2)^2-(-1)^{\frac{p-1}{2}}(8E_{p-3}-4E_{2p-4})$$
$$+p\left(-9q_p(2)^3+6(-1)^{\frac{p-1}{2}}q_p(2)E_{p-3}-\frac{14}{3}B_{p-3}\right)\pmod{p^2}.$$

By Corollary 3.4 we have $\sum_{i=1}^{[p/4]}\frac{1}{i^3}\equiv -9B_{p-3}\pmod p$. Thus,

$$\begin{aligned}&\left(\sum_{i=1}^{[p/4]}\frac{1}{i}\right)^3-3\left(\sum_{i=1}^{[p/4]}\frac{1}{i}\right)\left(\sum_{i=1}^{[p/4]}\frac{1}{i^2}\right)+2\sum_{i=1}^{[p/4]}\frac{1}{i^3}\\&\equiv\left(-3q_p(2)\right)^3-3\left(-3q_p(2)\right)\cdot 4(-1)^{\frac{p-1}{2}}E_{p-3}+2(-9B_{p-3})\\&=6\left(-\frac{9}{2}q_p(2)^3+6(-1)^{\frac{p-1}{2}}q_p(2)E_{p-3}-3B_{p-3}\right)\pmod p.\end{aligned}$$

Now putting all the above together with Lemma 2.9 and the fact $E_{2p-4}\equiv E_{p-3}\pmod p$ yields the result. □

Remark 3.3. The congruence $(-1)^{[\frac{p}{4}]}\binom{p-1}{[\frac{p}{4}]}\equiv 1+3pq_p(2)\pmod{p^2}$ was known to Lehmer. See [L, (51)].

For any prime $p>3$ we recall the Legendre symbol

$$\left(\frac{p}{3}\right)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 3,\\ -1 & \text{if } p\equiv 2\pmod 3.\end{cases}$$

Theorem 3.9. *Let p be a prime greater than 5. Then*

(i) $\sum_{k=1}^{[p/6]}\frac{1}{k^2}\equiv 5\sum_{k=1}^{[p/3]}\frac{1}{k^2}\equiv\frac{1}{2}(\frac{p}{3})B_{p-2}(\frac{1}{6})\pmod p$.

(ii) $\sum_{k=1}^{[p/6]}\frac{1}{k}\equiv -2q_p(2)-\frac{3}{2}q_p(3)+p(q_p(2)^2+\frac{3}{4}q_p(3)^2)-\frac{p}{12}(\frac{p}{3})B_{p-2}(\frac{1}{6})\pmod{p^2}$.

(iii) $\sum_{k=1}^{[p/3]}\frac{1}{k}\equiv -\frac{3}{2}q_p(3)+\frac{3}{4}pq_p(3)^2-\frac{p}{30}(\frac{p}{3})B_{p-2}(\frac{1}{6})\pmod{p^2}$.

(iv) $\sum_{k=1}^{[2p/3]}\frac{(-1)^{k-1}}{k}\equiv 9\sum_{k=1,\,3|k+p}^{p-1}\frac{1}{k}\equiv\frac{p}{10}(\frac{p}{3})B_{p-2}(\frac{1}{6})\pmod{p^2}$.

(v) *We have*

$$(-1)^{[\frac{p}{6}]}\binom{p-1}{[\frac{p}{6}]}\equiv 1+p\left(2q_p(2)+\frac{3}{2}q_p(3)\right)+p^2\left(q_p(2)^2+3q_p(2)q_p(3)\right.$$
$$\left.+\frac{3}{8}q_p(3)^2-\frac{1}{6}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right)\right)\pmod{p^3}$$

and

$$(-1)^{[\frac{p}{3}]}\binom{p-1}{[\frac{p}{3}]} \equiv 1+\frac{3}{2}pq_p(3)+\frac{3}{8}p^2q_p(3)^2-\frac{p^2}{60}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right) \pmod{p^3}.$$

Proof. Taking $k=2$ and $m=3,6$ in Theorem 3.3 and using Lemma 2.6 we see that

$$\sum_{k=1}^{[p/6]}\frac{1}{k^2} \equiv \frac{B_{p-2}(\{\frac{-p}{6}\})}{p-2} = -\left(\frac{p}{3}\right)\frac{B_{p-2}(\frac{1}{6})}{p-2} \equiv \frac{1}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right) \pmod p$$

and

$$\sum_{k=1}^{[p/3]}\frac{1}{k^2} \equiv \frac{B_{p-2}(\{\frac{-p}{3}\})}{p-2} = -\left(\frac{p}{3}\right)\frac{B_{p-2}(\frac{1}{3})}{p-2} \equiv \frac{1}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod p.$$

By Raabe's theorem (cf. [S2, Lemma 2.2]) we have $B_{p-2}(\frac{1}{6})+B_{p-2}(\frac{1}{6}+\frac{1}{2})=2^{1-(p-2)}B_{p-2}(\frac{1}{3})$. Thus

$$B_{p-2}\left(\frac{1}{6}\right)=2^{3-p}B_{p-2}\left(\frac{1}{3}\right)-B_{p-2}\left(\frac{2}{3}\right)=\left(2^{3-p}+1\right)B_{p-2}\left(\frac{1}{3}\right)\equiv 5B_{p-2}\left(\frac{1}{3}\right) \pmod p.$$

Hence (i) holds.

Suppose $m\in\{3,6\}$. Taking $k=1$, $r=0$ and $m=3,6$ in Lemma 2.7 and using Lemma 2.6 we see that

$$\begin{aligned}\sum_{k=1}^{[\frac{p}{m}]}\frac{1}{k} &= m\sum_{\substack{x=1\\ x\equiv 0 \pmod m}}^{p-1}\frac{1}{x} \equiv \frac{B_{\varphi(p^3)}(\{\frac{-p}{m}\})-B_{\varphi(p^3)}}{\varphi(p^3)}-\frac{p}{m}\cdot\frac{B_{p-2}(\{\frac{-p}{m}\})}{p-2}\\ &= \frac{B_{\varphi(p^3)}(\frac{1}{m})-B_{\varphi(p^3)}}{\varphi(p^3)}+\frac{p}{m}\left(\frac{p}{3}\right)\frac{B_{p-2}(\frac{1}{m})}{p-2} \pmod{p^2}.\end{aligned}$$

From the proof of Theorem 3.1 we have

$$\frac{B_{\varphi(p^3)}-B_{\varphi(p^3)}(\frac{1}{m})}{\varphi(p^3)} \equiv \begin{cases}\frac{3}{2}q_p(3)-\frac{3}{4}pq_p(3)^2 \pmod{p^2} & \text{if } m=3,\\ 2q_p(2)-pq_p(2)^2+\frac{3}{2}q_p(3)-\frac{3}{4}pq_p(3)^2 \pmod{p^2} & \text{if } m=6.\end{cases}$$

Hence (ii) and (iii) follow from the above and the fact $B_{p-2}(\frac{1}{3})\equiv\frac{1}{5}B_{p-2}(\frac{1}{6}) \pmod p$.

Now we consider (iv). Since

$$\sum_{1\leqslant k<\frac{2p}{3}}\frac{1-(-1)^{k-1}}{k}=\sum_{\substack{1\leqslant k<\frac{2p}{3}\\ 2\mid k}}\frac{2}{k}=\sum_{1\leqslant k<\frac{p}{3}}\frac{1}{k} \quad\text{and}\quad \sum_{1\leqslant k<\frac{p}{2}}\frac{1}{k^2}\equiv 0 \pmod p$$

by [S3, Corollary 5.2], using (i) we see that

$$\begin{aligned}\sum_{1\leqslant k<\frac{2p}{3}}\frac{(-1)^{k-1}}{k}&=\sum_{1\leqslant k<\frac{2p}{3}}\frac{1}{k}-\sum_{1\leqslant k<\frac{p}{3}}\frac{1}{k}=\sum_{\frac{p}{3}<k<\frac{2p}{3}}\frac{1}{k}\\&=\sum_{\frac{p}{3}<k<\frac{p}{2}}\left(\frac{1}{k}+\frac{1}{p-k}\right)=\sum_{\frac{p}{3}<k<\frac{p}{2}}\frac{p}{k(p-k)}\\&\equiv -p\sum_{\frac{p}{3}<k<\frac{p}{2}}\frac{1}{k^2}\equiv p\left(\sum_{1\leqslant k<\frac{p}{2}}\frac{1}{k^2}-\sum_{\frac{p}{3}<k<\frac{p}{2}}\frac{1}{k^2}\right)\\&=p\sum_{1\leqslant k<\frac{p}{3}}\frac{1}{k^2}\equiv\frac{p}{10}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right)\pmod{p^2}.\end{aligned}$$

On the other hand, noting that $\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0 \pmod{p^2}$ (cf. [L,S3]) and using (iii) and Theorem 3.1(i) we see that

$$\begin{aligned}\sum_{\substack{k=1\\3\mid k+p}}^{p-1}\frac{1}{k}&=\sum_{k=1}^{p-1}\frac{1}{k}-\sum_{\substack{k=1\\k\equiv 0\ (\mathrm{mod}\ 3)}}^{p-1}\frac{1}{k}-\sum_{\substack{k=1\\k\equiv p\ (\mathrm{mod}\ 3)}}^{p-1}\frac{1}{k}\\&\equiv-\frac{1}{3}\sum_{k=1}^{[\frac{p}{3}]}\frac{1}{k}-\sum_{\substack{k=1\\k\equiv p\ (\mathrm{mod}\ 3)}}^{p-1}\frac{1}{k}\\&\equiv-\frac{1}{3}\left(-\frac{3}{2}q_p(3)+\frac{3}{4}pq_p(3)^2-\frac{p}{30}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right)\right)-\left(\frac{1}{2}q_p(3)-\frac{1}{4}pq_p(3)^2\right)\\&=\frac{p}{90}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right)\pmod{p^2}.\end{aligned}$$

Thus (iv) is true. Finally we consider (v). By Lemma 2.9, for $m=3,6$ we have

$$(-1)^{[\frac{p}{m}]}\binom{p-1}{[\frac{p}{m}]}\equiv 1-p\sum_{k=1}^{[\frac{p}{m}]}\frac{1}{k}+\frac{p^2}{2}\left(\left(\sum_{k=1}^{[\frac{p}{m}]}\frac{1}{k}\right)^2-\sum_{k=1}^{[\frac{p}{m}]}\frac{1}{k^2}\right)\pmod{p^3}.$$

Thus appealing to (i)–(iii) we obtain

$$\begin{aligned}&(-1)^{[\frac{p}{6}]}\binom{p-1}{[\frac{p}{6}]}\\&\quad\equiv 1-p\left(-2q_p(2)-\frac{3}{2}q_p(3)+p\left(q_p(2)^2+\frac{3}{4}q_p(3)^2\right)-\frac{p}{12}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right)\right)\\&\qquad+\frac{p^2}{2}\left(\left(-2q_p(2)-\frac{3}{2}q_p(3)\right)^2-\frac{1}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right)\right)\pmod{p^3}\end{aligned}$$

and

$$(-1)^{[\frac{p}{3}]}\binom{p-1}{[\frac{p}{3}]} \equiv 1 - p\left(-\frac{3}{2}q_p(3) + \frac{3}{4}pq_p(3)^2 - \frac{p}{30}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right)\right) + \frac{p^2}{2}\left(\left(-\frac{3}{2}q_p(3)\right)^2 - \frac{1}{10}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right)\right) \pmod{p^3}.$$

This yields (v) and hence the theorem is proved. □

Remark 3.4. Let $p > 5$ be a prime. The congruences

$$\sum_{k=1}^{[p/3]} \frac{1}{k} \equiv -\frac{3}{2}q_p(3) \pmod p, \qquad \sum_{k=1}^{[p/6]} \frac{1}{k} \equiv -2q_p(2) - \frac{3}{2}q_p(3) \pmod p,$$

$$(-1)^{[p/3]}\binom{p-1}{[p/3]} \equiv 1 + \frac{3}{2}pq_p(3) \pmod{p^2},$$

$$(-1)^{[p/6]}\binom{p-1}{[p/6]} \equiv 1 + 2pq_p(2) + \frac{3}{2}pq_p(3) \pmod{p^2}$$

were known to Lehmer [L], and the congruence

$$\sum_{k=1}^{[p/3]} \frac{1}{k^2} \equiv \frac{1}{5}\sum_{k=1}^{[p/6]} \frac{1}{k^2} \pmod p$$

is due to Schwindt (cf. [R,L]). In [S1], using the formulas for $\sum_{k\equiv r \pmod 3}\binom{p}{k}$ the author proved that

$$\sum_{k=1}^{[\frac{2p}{3}]} \frac{(-1)^{k-1}}{k} \equiv 0 \pmod p.$$

Corollary 3.10. *Let $p > 5$ be a prime. Then*

(i) $\sum_{k=1}^{[p/6]} \frac{1}{k} + \frac{p}{6}\sum_{k=1}^{[p/6]} \frac{1}{k^2} \equiv -2q_p(2) - \frac{3}{2}q_p(3) + p(q_p(2)^2 + \frac{3}{4}q_p(3)^2) \pmod{p^2}$.

(ii) $\sum_{k=1}^{[p/3]} \frac{1}{k} + \frac{p}{3}\sum_{k=1}^{[p/3]} \frac{1}{k^2} \equiv -\frac{3}{2}q_p(3) + \frac{3}{4}pq_p(3)^2 \pmod{p^2}$.

(iii) *We have*

$$(-1)^{[\frac{p}{6}]}\binom{p-1}{[\frac{p}{6}]} - 10(-1)^{[\frac{p}{3}]}\binom{p-1}{[\frac{p}{3}]} \equiv -9 + p\left(2q_p(2) - \frac{27}{2}q_p(3)\right) + p^2\left(q_p(2)^2 + 3q_p(2)q_p(3) - \frac{27}{8}q_p(3)^2\right) \pmod{p^3}.$$

Corollary 3.11. *Let $p>5$ be a prime. Then*

(i)
$$\sum_{1\leqslant k<\frac{2p}{3}}\frac{(-1)^{k-1}}{k}\equiv 9\sum_{\substack{1\leqslant k<p\\ 3\mid k+p}}\frac{1}{k}\equiv\frac{p}{5}\sum_{k=1}^{[\frac{p}{6}]}\frac{1}{k^2}\pmod{p^2},$$

(ii)
$$\sum_{1\leqslant k<\frac{2p}{3}}\frac{(-1)^{k-1}}{k}+3\sum_{1\leqslant k<\frac{p}{3}}\frac{1}{k}\equiv-\frac{9}{2}q_p(3)+\frac{9}{4}pq_p(3)^2\pmod{p^2},$$

(iii)
$$\sum_{1\leqslant k<\frac{2p}{3}}\frac{(-1)^{k-1}}{k}-2\sum_{\frac{p}{6}<k<\frac{p}{3}}\frac{1}{k}\equiv-4q_p(2)+2pq_p(2)^2\pmod{p^2},$$

(iv)
$$3\sum_{1\leqslant k<\frac{p}{6}}\frac{1}{k}+5\sum_{\frac{p}{6}<k<\frac{p}{3}}\frac{1}{k}\equiv 4q_p(2)-\frac{9}{2}q_p(3)+p\left(-2q_p(2)^2+\frac{9}{4}q_p(3)^2\right)\pmod{p^2}.$$

4. Congruences for $\sum_{k=1}^{p-1}\frac{2^k}{k}$ and $\sum_{k=1}^{p-1}\frac{2^k}{k^2}$

Let $p>3$ be a prime. For $n\in\mathbb{N}$ let

$$G_n(x)=\sum_{k=1}^{p-1}\frac{x^k}{k^n}.$$

Then $G_n(x)\in\mathbb{Z}_p[x]$. In [Gl] Glaisher showed that

$$G_1(2)=\sum_{k=1}^{p-1}\frac{2^k}{k}\equiv-2q_p(2)\pmod{p}. \tag{4.1}$$

In 2004 Granville [Gr] proved the following Skula's conjecture:

$$G_2(2)=\sum_{k=1}^{p-1}\frac{2^k}{k^2}\equiv-q_p(2)^2\pmod{p}. \tag{4.2}$$

In the section we determine $G_1(2)\pmod{p^3}$ and $G_2(2)\pmod{p^2}$.

Lemma 4.1. *Let p be an odd prime. In $\mathbb{Z}_p[x]$ we have*

$$G_2(x)\equiv\frac{1}{p}\left(\frac{1+(x-1)^p-x^p}{p}-\sum_{k=1}^{p-1}\frac{(1-x)^k-1}{k}\right)+p\sum_{r=2}^{p-1}\frac{x^r}{r^2}\sum_{s=1}^{r-1}\frac{1}{s}\pmod{p^2}.$$

Proof. From Lemma 2.9 we know that

$$(-1)^k\binom{p-1}{k}\equiv 1-p\sum_{s=1}^{k}\frac{1}{s}\pmod{p^2}$$

for $k\leqslant p-1$. Thus

$$\sum_{r=1}^{p-1}\frac{x^r}{r^2}\left(1-p\sum_{s=1}^{r-1}\frac{1}{s}\right)\equiv\sum_{r=1}^{p-1}\frac{x^r}{r^2}(-1)^{r-1}\binom{p-1}{r-1}=\frac{1}{p}\sum_{r=1}^{p-1}\frac{(-1)^{r-1}}{r}\binom{p}{r}x^r\pmod{p^2}.$$

Since $\int_0^1 t^{r-1}\,dt=\frac{1}{r}$, setting $y=1-xt$ we see that

$$\begin{aligned}
&\sum_{r=1}^{p-1}\frac{(-1)^{r-1}}{r}\binom{p}{r}x^r\\
&\quad=\int_0^1\sum_{r=1}^{p-1}(-1)^{r-1}\binom{p}{r}x^r t^{r-1}\,dt=\int_0^1\frac{(1-xt)^p-1-(-xt)^p}{-t}\,dt\\
&\quad=-\int_1^{1-x}\frac{y^p-1-(y-1)^p}{y-1}\,dy=\int_1^{1-x}\left((y-1)^{p-1}-\sum_{k=1}^{p}y^{k-1}\right)dy\\
&\quad=\left(\frac{(y-1)^p}{p}-\sum_{k=1}^{p}\frac{y^k}{k}\right)\Bigg|_1^{1-x}=-\frac{x^p}{p}-\sum_{k=1}^{p}\frac{(1-x)^k}{k}+\sum_{k=1}^{p}\frac{1}{k}\\
&\quad=\frac{1-x^p+(x-1)^p}{p}-\sum_{k=1}^{p-1}\frac{(1-x)^k-1}{k}.
\end{aligned}$$

Now combining the above we obtain the result. □

Lemma 4.2. *Let $p>3$ be a prime. Then*

$$G_2(2)\equiv -q_p(2)^2+p\left(\sum_{r=2}^{p-1}\frac{2^r}{r^2}\sum_{s=1}^{r-1}\frac{1}{s}+\frac{2}{3}q_p(2)^3-\frac{1}{12}B_{p-3}\right)\pmod{p^2}.$$

Proof. From [S3, Remark 5.3] we know that

$$\sum_{k=1}^{p-1}\frac{1-(-1)^k}{k}=2\sum_{\substack{k=1\\ 2\nmid k}}^{p-1}\frac{1}{k}\equiv 2q_p(2)-pq_p(2)^2+\frac{2}{3}p^2q_p(2)^3-\frac{1}{12}p^2B_{p-3}\pmod{p^3}.$$

Thus putting $x=2$ in Lemma 4.1 and applying the above gives the result. □

Lemma 4.3. *Let $p > 3$ be a prime and $n \in \mathbb{N}$. Then*

$$npG_{n+1}(x) \equiv (-1)^n x^p G_n(1/x) - G_n(x) \pmod{p^2}.$$

Proof. Clearly we have

$$\begin{aligned}
x^p G_n\left(\frac{1}{x}\right) &= \sum_{i=1}^{p-1} \frac{x^p}{x^i \cdot i^n} = \sum_{k=1}^{p-1} \frac{x^k}{(p-k)^n} \equiv \sum_{k=1}^{p-1} \frac{x^k}{(-k)^n + n(-k)^{n-1}p} \\
&= \sum_{k=1}^{p-1} \frac{(k+np)x^k}{(-k)^{n-1}(n^2p^2-k^2)} \equiv (-1)^n \sum_{k=1}^{p-1} \frac{(k+np)x^k}{k^{n+1}} \\
&= (-1)^n \big(npG_{n+1}(x) + G_n(x)\big) \pmod{p^2}.
\end{aligned}$$

This yields the result. □

Theorem 4.1. *Let $p > 3$ be a prime. Then*

(i) $$\sum_{k=1}^{p-1} \frac{2^k}{k} \equiv -2q_p(2) - \frac{7}{12}p^2 B_{p-3} \pmod{p^3}.$$

(ii) $$\sum_{k=1}^{p-1} \frac{2^k}{k^2} \equiv -q_p(2)^2 + p\left(\frac{2}{3}q_p(2)^3 + \frac{7}{6}B_{p-3}\right) \pmod{p^2}.$$

(iii) $$\sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \equiv q_p(2) - \frac{p}{2}q_p(2)^2 \pmod{p^2}.$$

(iv) $$\sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k} \equiv -\frac{1}{2}q_p(2)^2 \pmod{p}.$$

Proof. Suppose $x \in \mathbb{Z}_p$ and $x \not\equiv 0 \pmod{p}$. From Lemma 4.1 we see that

$$\begin{aligned}
x^p G_2(1/x) &\equiv \frac{1}{p}\left(\frac{x^p + (1-x)^p - 1}{p} + \sum_{k=1}^{p-1} \frac{x^k - (x-1)^k}{k} \cdot x^{p-k}\right) \\
&\quad + p \sum_{r=1}^{p-1} \frac{x^{p-r}}{r^2} \sum_{s=1}^{r-1} \frac{1}{s} \pmod{p^2}.
\end{aligned}$$

As

$$\sum_{s=1}^{p-1-k} \frac{1}{s} = \sum_{s=1}^{p-1} \frac{1}{s} - \sum_{r=1}^{k} \frac{1}{p-r} \equiv 0 - \sum_{r=1}^{k} \frac{1}{p-r} \equiv \sum_{r=1}^{k} \frac{1}{r} \pmod{p}, \tag{4.3}$$

we see that

$$\sum_{r=1}^{p-1}\frac{x^{p-r}}{r^2}\sum_{s=1}^{r-1}\frac{1}{s}=\sum_{k=1}^{p-1}\frac{x^k}{(p-k)^2}\sum_{s=1}^{p-k-1}\frac{1}{s}\equiv\sum_{k=1}^{p-1}\frac{x^k}{k^2}\sum_{s=1}^{k}\frac{1}{s}\pmod p.$$

Thus

$$\begin{aligned}x^pG_2(1/x)\equiv{}&\frac{1}{p}\left(\frac{x^p-(x-1)^p-1}{p}+\sum_{k=1}^{p-1}\frac{x^p-x^{p-k}(x-1)^k}{k}\right)\\&+p\sum_{k=1}^{p-1}\frac{x^k}{k^2}\sum_{s=1}^{k}\frac{1}{s}\pmod{p^2}.\end{aligned}$$

Hence applying Lemmas 4.1 and 4.3 we see that

$$\begin{aligned}2pG_3(x)&\equiv x^pG_2(1/x)-G_2(x)\\&\equiv\frac{1}{p}\left\{\frac{2(x^p-(x-1)^p-1)}{p}+\sum_{k=1}^{p-1}\left(\frac{x^p-x^{p-k}(x-1)^k}{k}+\frac{(1-x)^k-1}{k}\right)\right\}\\&\quad+p\sum_{k=1}^{p-1}\frac{x^k}{k^2}\left(\sum_{s=1}^{k}\frac{1}{s}-\sum_{s=1}^{k-1}\frac{1}{s}\right)\pmod{p^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}G_3(x)\equiv\frac{1}{p^2}\Bigg\{&\frac{2(x^p-(x-1)^p-1)}{p}\\&+\sum_{k=1}^{p-1}\frac{1}{k}\big(x^p-1+(1-x)^k-x^{p-k}(x-1)^k\big)\Bigg\}\pmod p.\end{aligned}\tag{4.4}$$

Taking $x=2$ we find

$$G_3(2)\equiv\frac{1}{p^2}\left\{4q_p(2)+\big(2^p-2\big)\sum_{k=1}^{p-1}\frac{1}{k}+\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k}-2^p\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}\right\}\pmod p.$$

It is well known that $\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0\pmod{p^2}$ (cf. [L, p. 353]). From [S3, Theorem 5.2(c)] we also know that

$$\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k}=\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k}\equiv-2q_p(2)+pq_p(2)^2-\frac{2}{3}p^2q_p(2)^3-\frac{7}{12}p^2B_{p-3}\pmod{p^3}.$$

Thus

$$G_3(2) \equiv \frac{1}{p^2}\left\{2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{7}{12}p^2B_{p-3} - 2^p\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}\right\} \pmod p. \quad (4.5)$$

As

$$\begin{aligned}-\sum_{k=1}^{p-1}\frac{2^{p-k}}{k} &= -\sum_{k=1}^{p-1}\frac{2^k}{p-k} = -\sum_{k=1}^{p-1}\frac{2^k(p^2+kp+k^2)}{p^3-k^3} \equiv \sum_{k=1}^{p-1}\frac{2^k(p^2+kp+k^2)}{k^3} \\ &= p^2G_3(2) + pG_2(2) + G_1(2) \pmod{p^3},\end{aligned} \quad (4.6)$$

by (4.5) we have

$$\begin{aligned}p^2G_3(2) \equiv{}& 2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{7}{12}p^2B_{p-3} \\ &+ p^2G_3(2) + pG_2(2) + G_1(2) \pmod{p^3}.\end{aligned}$$

Namely,

$$pG_2(2) \equiv -G_1(2) - 2q_p(2) - pq_p(2)^2 + \frac{2}{3}p^2q_p(2)^3 + \frac{7}{12}p^2B_{p-3} \pmod{p^3}. \quad (4.7)$$

According to Lemma 4.1 we have

$$G_2(-1) \equiv \frac{1}{p}\left(-2q_p(2) - \sum_{k=1}^{p-1}\frac{2^k}{k} + \sum_{k=1}^{p-1}\frac{1}{k}\right) + p\sum_{r=1}^{p-1}\frac{(-1)^r}{r^2}\sum_{s=1}^{r-1}\frac{1}{s} \pmod{p^2}.$$

As $p > 3$, by [S3, Corollary 5.2 and Theorem 5.1] or [L] we have

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k^3} \equiv -2B_{p-3} \pmod p \quad \text{and} \quad \sum_{k=1}^{p-1}\frac{1}{k} \equiv -\frac{1}{3}p^2B_{p-3} \pmod{p^3}.$$

Thus applying (4.3) we see that

$$\begin{aligned}\sum_{r=1}^{p-1}\frac{(-1)^r}{r^2}\sum_{s=1}^{r-1}\frac{1}{s} &= \sum_{k=1}^{\frac{p-1}{2}}\frac{1}{(2k)^2}\sum_{s=1}^{2k-1}\frac{1}{s} - \sum_{k=1}^{\frac{p-1}{2}}\frac{1}{(p-2k)^2}\sum_{s=1}^{p-2k-1}\frac{1}{s} \\ &\equiv \sum_{k=1}^{\frac{p-1}{2}}\frac{1}{(2k)^2}\left(\sum_{s=1}^{2k-1}\frac{1}{s} - \sum_{s=1}^{2k}\frac{1}{s}\right) = -\frac{1}{8}\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k^3} \equiv \frac{1}{4}B_{p-3} \pmod p\end{aligned}$$

and hence

$$G_2(-1) \equiv \frac{1}{p}\left(-2q_p(2) - \sum_{k=1}^{p-1}\frac{2^k}{k} - \frac{1}{3}p^2B_{p-3}\right) + \frac{p}{4}B_{p-3} \pmod{p^2}.$$

On the other hand, using [S3, Corollaries 5.1 and 5.2] we have

$$G_2(-1)=\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}=\sum_{\substack{k=1\\2\mid k}}^{p-1}\frac{2}{k^2}-\sum_{k=1}^{p-1}\frac{1}{k^2}=\frac{1}{2}\sum_{i=1}^{\frac{p-1}{2}}\frac{1}{i^2}-\sum_{k=1}^{p-1}\frac{1}{k^2}$$
$$\equiv\frac{1}{2}\cdot\frac{2(2^3-1)}{2(2+1)}pB_{p-3}-\frac{2}{3}pB_{p-3}=\frac{1}{2}pB_{p-3}\pmod{p^2}.$$

Hence

$$-2q_p(2)-\sum_{k=1}^{p-1}\frac{2^k}{k}-\frac{1}{3}p^2B_{p-3}+\frac{p^2}{4}B_{p-3}\equiv pG_2(-1)\equiv\frac{p^2}{2}B_{p-3}\pmod{p^3}.$$

This yields

$$G_1(2)=\sum_{k=1}^{p-1}\frac{2^k}{k}\equiv-2q_p(2)-\frac{7}{12}p^2B_{p-3}\pmod{p^3}.$$

So (i) holds. Substituting this into (4.7) gives

$$pG_2(2)\equiv-pq_p(2)^2+\frac{7}{6}p^2B_{p-3}+\frac{2}{3}p^2q_p(2)^3\pmod{p^3}.$$

That is,

$$G_2(2)\equiv-q_p(2)^2+p\left(\frac{2}{3}q_p(2)^3+\frac{7}{6}B_{p-3}\right)\pmod{p^2}.$$

Thus (ii) is true. By (4.6) we have

$$\sum_{k=1}^{p-1}\frac{2^{p-k}}{k}\equiv-pG_2(2)-G_1(2)\pmod{p^2}.$$

From the above we know that

$$G_1(2)\equiv-2q_p(2)\pmod{p^2}\quad\text{and}\quad G_2(2)\equiv-q_p(2)^2\pmod{p}.$$

Thus

$$\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}\equiv\frac{-G_1(2)-pG_2(2)}{2^p}\equiv\frac{2q_p(2)+pq_p(2)^2}{2(1+pq_p(2))}$$
$$\equiv\left(q_p(2)+\frac{p}{2}q_p(2)^2\right)\bigl(1-pq_p(2)\bigr)\equiv q_p(2)-\frac{p}{2}q_p(2)^2\pmod{p^2}.$$

This proves (iii). From Lemma 4.3 we have

$$2pG_3(2) \equiv 2^p G_2(1/2) - G_2(2) \pmod{p^2}.$$

Thus,

$$G_2\left(\frac{1}{2}\right) \equiv \frac{G_2(2)}{2^p} \equiv \frac{-q_p(2)^2}{2} \pmod{p}.$$

This proves (iv) and hence the theorem is proved. □

Corollary 4.1. *Let $p > 3$ be a prime. Then*

$$q_p(2) \equiv -\sum_{k=1}^{p-1} \frac{2^{k-1}}{k} \pmod{p^2}.$$

Remark 4.1. Let $p > 3$ be a prime. By [DS, (5)] and [S3, Corollary 5.2(b)] we have

$$G_3(2) = \sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv -\frac{1}{3} q_p(2)^3 + \frac{7}{48} \sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j^3} \equiv -\frac{1}{3} q_p(2)^3 - \frac{7}{24} B_{p-3} \pmod{p}.$$

This together with (4.5) and Corollary 3.1 yields

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} &\equiv \frac{q_p(2) + \frac{1}{2} p q_p(2)^2 - \frac{1}{6} p^2 q_p(2)^3 - \frac{7}{48} p^2 B_{p-3}}{1 + p q_p(2)} \\
&\equiv q_p(2) - \frac{1}{2} p q_p(2)^2 + \frac{1}{3} p^2 q_p(2)^3 - \frac{7}{48} p^2 B_{p-3} \\
&\equiv 4 \sum_{\substack{k=1 \\ 4 \mid k+p}}^{p-1} \frac{1}{k} \pmod{p^3}.
\end{aligned}$$

By the proof of Theorem 4.1, we also have

$$\begin{aligned}
&\sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k} = G_2\left(\frac{1}{2}\right) \equiv \frac{pG_3(2) + G_2(2)/2}{2^{p-1}} \\
&\quad \equiv \bigl(1 - p q_p(2)\bigr)\left(-\frac{1}{3} p q_p(2)^3 - \frac{7}{24} p B_{p-3} + \frac{1}{2}\left(-q_p(2)^2 + \frac{2}{3} p q_p(2)^3 + \frac{7}{6} p B_{p-3}\right)\right) \\
&\quad \equiv -\frac{1}{2} q_p(2)^2 + \frac{1}{2} p q_p(2)^3 + \frac{7}{24} p B_{p-3} \pmod{p^2}.
\end{aligned}$$

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